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On certain $\mathbf{SL}(2,\mathbf{R})\times\mathbf{SL}(2,\mathbf{R})$-actions on S^3*

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0 Introduction

There are two types of smooth actions of $K = \mathbf{SO}(2)\times\mathbf{SO}(2)$ on S^3, each of which has principal orbits of codimension one. In this paper, we shall study actions of $G = \mathbf{SL}(2,\mathbf{R})\times\mathbf{SL}(2,\mathbf{R})$ on S^3 which is regarded as an extension of the K-action. Notice that K is the maximal compact subgroup of G. As a result, we can state : for each positive integer r, there are uncountably many topologically distinct continuous G-actions on S^3, the number of open orbits of which is $r+2$. There is a little difference in the results between two types.

1 Standard actions of $\mathbf{SL}(2,\mathbf{R})\times\mathbf{SL}(2,\mathbf{R})$ on S^3

Define a smooth action ψ of $\mathbf{SL}(2,\mathbf{R})\times\mathbf{SL}(2,\mathbf{R})$ on S^3 by

$$\psi((A,B),X) = \|AXB^{-1}\|^{-1}AXB^{-1}$$

for $A,B\in\mathbf{SL}(2,\mathbf{R})$ and $X\in S^3$. Denote by ψ_0 the restricted action of ψ to the compact group $\mathbf{SO}(2)\times\mathbf{SO}(2)$. We call ψ,ψ_0 the standard actions of the first type. Here we identify S^3 with the unit sphere of $M_2(\mathbf{R})$ which is the vector space of all real 2×2 matrices with the norm

$$\|X\| = (\mathrm{trace}\,{}^tXX)^{1/2} \quad \text{for} \quad X\in M_2(\mathbf{R}).$$

Denote by E_{11},E_{12},E_{21} and E_{22}, the matrix units of $M_2(\mathbf{R})$. Put $I_2 = E_{11}+E_{22}$ and $J = E_{11}-E_{22}$.

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Lemma 1.1. *Let $X \in S^3$.*

(1) *There exist $A, B \in \mathbf{SO}(2) : AXB^{-1} = pE_{11} + qE_{22}$ for certain real numbers p, q with $p \geq |q|$ and $p^2 + q^2 = 1$.*

(2) *If* $\mathrm{rank} X = 2$, *then there exist $A, B \in \mathbf{SL}(2, \mathbf{R})$ and $c > 0 : AXB^{-1} = c(E_{11} + \varepsilon E_{22})/\sqrt{2}$, where $\varepsilon = 1$ for $\det X > 0$ and $\varepsilon = -1$ for $\det X < 0$. If* $\mathrm{rank} X = 1$, *then there exist $A, B \in \mathbf{SL}(2, \mathbf{R}) : AXB^{-1} = E_{11}$.*

Proof. Denote by X_1, X_2 the first and the second columns of $X \in M_2(\mathbf{R})$ respectively, that is, $X = [X_1, X_2]$. Put

$$R(\tau) = \begin{bmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{bmatrix}.$$

Then we see

$$XR(\tau)^{-1} = [X_1', X_2'] = [\cos\tau X_1 + \sin\tau X_2, -\sin\tau X_1 + \cos\tau X_2].$$

Therefore, there exists τ such that $X_1' \perp X_2'$ and $\|X_1'\| \geq \|X_2'\|$. Put $B = R(\tau)$. Then, we obtain $AXB^{-1} = pE_{11} + qE_{22}$ for certain $A \in \mathbf{SO}(2)$.

To show the second part, we may assume $X = pE_{11} + qE_{22}$, where $p \geq |q|$. Suppose $\mathrm{rank} X = 2$. Then there exists a diagonal matrix D of $\mathbf{SL}(2, \mathbf{R})$ such that $DX = c(E_{11} + \varepsilon E_{22})/\sqrt{2}$. Suppose $\mathrm{rank} X = 1$. Then $q = 0$, and hence $X = E_{11}$. q.e.d.

By this lemma, we see that the standard action ψ of $\mathbf{SL}(2, \mathbf{R}) \times \mathbf{SL}(2, \mathbf{R})$ on S^3 has just three orbits. Two of them are open orbits through $(E_{11} + \varepsilon E_{22})/\sqrt{2}$ for $\varepsilon = \pm 1$, respectively, and the other is a compact orbit through E_{11}. In particular, two singular orbits of restricted action ψ_0 of $\mathbf{SO}(2) \times \mathbf{SO}(2)$ on S^3 are contained in open orbits of the action ψ, respectively.

By direct calculation, we obtain the condition that an element $(A, B) \in \mathbf{SO}(2) \times \mathbf{SO}(2)$ belongs to the isotropy subgroup at $pE_{11} + qE_{22}$, where $p \geq |q|$ and $p^2 + q^2 = 1$. In fact, if $p > |q| > 0$, then $A = B = \varepsilon I_2$, where $\varepsilon = \pm 1$. If $p = q = 1/\sqrt{2}$, then $A = B$, and if $p = -q = 1/\sqrt{2}$, then $B = A^{-1}$.

Moreover, we obtain the condition that an element $(A, B) \in \mathbf{SL}(2, \mathbf{R}) \times \mathbf{SL}(2, \mathbf{R})$ belongs to the isotropy subgroup at $pE_{11} + qE_{22}$, where $p \geq |q|$ and $p^2 + q^2 = 1$. In particular, if $p = q = 1/\sqrt{2}$, then $A = B$. If $p = -q = 1/\sqrt{2}$, then $B = JAJ^{-1}$. If $p = 1, q = 0$, then

$$A = \begin{bmatrix} a & * \\ 0 & * \end{bmatrix}, \quad B = \begin{bmatrix} b & 0 \\ * & * \end{bmatrix}, \quad ab > 0.$$

Define a smooth action φ of $\mathbf{SL}(2, \mathbf{R}) \times \mathbf{SL}(2, \mathbf{R})$ on S^3 by

$$\varphi((A, B), \mathbf{u} \oplus \mathbf{v}) = \|A\mathbf{u} \oplus B\mathbf{v}\|^{-1}(A\mathbf{u} \oplus B\mathbf{v})$$

for $A, B \in \mathbf{SL}(2,\mathbf{R})$ and $\mathbf{u}, \mathbf{v} \in \mathbf{R}^2$ such that $\|\mathbf{u}\oplus\mathbf{v}\| = 1$. Denote by φ_0 the restricted action of φ to the compact group $\mathbf{SO}(2)\times\mathbf{SO}(2)$. We call φ, φ_0 the standard actions of the second type.

Denote by $\mathbf{e}_1, \mathbf{e}_2$ the canonical basis of $\mathbf{R}^2$. We obtain the following result immediately.

Lemma 1.2. *Let $\mathbf{u}\oplus\mathbf{v} \in S^3$.*

(1) *There exist $A, B \in \mathbf{SO}(2) : A\mathbf{u}\oplus B\mathbf{v} = \|\mathbf{u}\|\mathbf{e}_1 \oplus \|\mathbf{v}\|\mathbf{e}_1$.*

(2) *If $\|\mathbf{u}\|\cdot\|\mathbf{v}\| \neq 0$, then there exist $A, B \in \mathbf{SL}(2,\mathbf{R}) : A\mathbf{u}\oplus B\mathbf{v} = (\mathbf{e}_1\oplus\mathbf{e}_1)/\sqrt{2}$.*

By this lemma, we see that the standard action φ of $\mathbf{SL}(2,\mathbf{R})\times\mathbf{SL}(2,\mathbf{R})$ on S^3 has just three orbits. One of them is an open orbit through $(\mathbf{e}_1\oplus\mathbf{e}_1)/\sqrt{2}$ and the others are compact orbits through $\mathbf{e}_1\oplus 0$ and $0\oplus\mathbf{e}_1$, respectively.

By direct calculation, we obtain the condition that an element $(A, B) \in \mathbf{SL}(2,\mathbf{R})\times\mathbf{SL}(2,\mathbf{R})$ belongs to the isotropy subgroup at $p\mathbf{e}_1\oplus q\mathbf{e}_1$, where $p \geq 0, q \geq 0$ and $p^2+q^2 = 1$. In particular, if $pq \neq 0$, then

$$A = \begin{bmatrix} a & * \\ 0 & * \end{bmatrix},\quad B = \begin{bmatrix} b & * \\ 0 & * \end{bmatrix};\quad a > 0, b > 0.$$

2 Certain closed subgroups of $\mathbf{SL}(2,\mathbf{R})\times\mathbf{SL}(2,\mathbf{R})$

Put $G = \mathbf{SL}(2,\mathbf{R})\times\mathbf{SL}(2,\mathbf{R})$, $K = \mathbf{SO}(2)\times\mathbf{SO}(2)$. Here we search closed subgroups of G which may be an isotropy subgroup at some point of S^3, with respect to an extended G-action of the standard K-action ψ_0 or φ_0.

Lemma 2.1. *Let S be a closed subgroup of G. Suppose S satisfies the following conditions.*

(1) $S\cap K = \{(\varepsilon I_2, \varepsilon I_2) | \varepsilon = \pm 1\}$, $\{(A, A) | A \in \mathbf{SO}(2)\}$ *or* $\{(A, A^{-1}) | A \in \mathbf{SO}(2)\}$,

(2) $\dim S \geq 3$.

Then, S is conjugate with one of the followings :

(a) $N_{2,2} = \left\{\left(\begin{bmatrix} a & * \\ 0 & * \end{bmatrix}, \begin{bmatrix} b & * \\ 0 & * \end{bmatrix}\right) \mid ab > 0\right\}$,

(b) $\{(A, A) | A \in \mathbf{SL}(2,\mathbf{R})\}$, $\{(A, JAJ^{-1}) | A \in \mathbf{SL}(2,\mathbf{R})\}$,

(c) $\left\{\left(\begin{bmatrix} a & * \\ 0 & * \end{bmatrix}, \begin{bmatrix} b & 0 \\ 0 & * \end{bmatrix}\right) \mid ab > 0\right\}$, $\left\{\left(\begin{bmatrix} a & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} b & * \\ 0 & * \end{bmatrix}\right) \mid ab > 0\right\}$,

(d) $S(\alpha,\beta,\gamma) = \left\{\left(\begin{bmatrix} \varepsilon e^{\alpha t-\beta\gamma k} & * \\ 0 & * \end{bmatrix}, \begin{bmatrix} \varepsilon e^{\beta t+\alpha\gamma k} & * \\ 0 & * \end{bmatrix}\right) \mid t \in \mathbf{R}, k \in \mathbf{Z}, \varepsilon = \pm 1\right\}$

for $(\alpha, \beta) \neq (0,0)$ *and* $\gamma \geq 0$.

Proof. Let S_i be the image of S by the natural projection of $G = \mathbf{SL}(2,\mathbf{R}) \times \mathbf{SL}(2,\mathbf{R})$ to the i-th factor, for $i = 1, 2$. Let $S_{(i)}$ be the closed subgroup of $\mathbf{SL}(2,\mathbf{R})$ such that

$$S_{(1)} \times \{I_2\} = S \cap (\mathbf{SL}(2,\mathbf{R}) \times \{I_2\}),\ \{I_2\} \times S_{(2)} = S \cap (\{I_2\} \times \mathbf{SL}(2,\mathbf{R})).$$

Then, $S_{(i)}$ is a normal subgroup of S_i, for $i = 1, 2$. Moreover, $S_{(1)} \times S_{(2)}$ is a normal subgroup of S, $S_1 \times S_2$ contains S as a subgroup, and there are natural isomorphisms

$$S_1/S_{(1)} \cong S_2/S_{(2)} \cong S/(S_{(1)} \times S_{(2)}).$$

By (1), we see $S_i \neq S_{(i)}$ for $i = 1, 2$. By (2), we may assume $\dim S_1 \geq 2$, without loss of generality. Since $-I_2 \in S_1$ by (1), we see S_1 is conjugate with one of the followings :

$$\mathbf{SL}(2,\mathbf{R}) \quad \text{or} \quad N(2) = \left\{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \right\}.$$

Denote by T^0 and $N(T)$ the identity component and the normalizer of a closed subgroup T of $\mathbf{SL}(2,\mathbf{R})$, respectively.

First, we assume $S_1 = \mathbf{SL}(2,\mathbf{R})$. Then, $S_{(1)} = \{I_2\}$ or C_2. Here, $C_2 = \{\pm I_2\}$ is the center of $\mathbf{SL}(2,\mathbf{R})$. Moreover, we see $S_2 = S_1$ and $S_{(2)} = S_{(1)}$. If $S_{(1)} = S_{(2)} = C_2$, then the corresponding S does not satisfy the condition (1). If $S_{(1)} = S_{(2)} = \{I_2\}$, then we see

$$S = \{(A, PAP^{-1}) | A \in \mathbf{SL}(2,\mathbf{R})\},$$

for some $P \in \mathbf{GL}(2,\mathbf{R})$. This is the case (b).

Next, we assume $S_1 = N(2)$. Then, $\dim S_{(1)} \leq 1$ or $S_{(1)} = N(2)^0$. Suppose $\dim S_{(1)} = 0$. Then we see $\dim S_2 = 2$ and $\dim S_{(2)} = 0$, and hence $\dim S = 2$, which contradicts to (2). Suppose $\dim S_{(1)} = 1$. Then we see $\dim S_2 = 2$ and $\dim S_{(2)} = 1$. Moreover, we can assume $S_1 = S_2 = N(2)$. Since $S_{(i)}$ is a normal subgroup of S_i, we see the identity component of $S_{(i)}$ coincides with $L(2)$. Here

$$L(2) = \left\{ \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \right\}.$$

Then $N(2)$ is the normalizer of $L(2)$. Consider the natural projection

$$\pi : N(2) \times N(2) \to N(2)/L(2) \times N(2)/L(2) \cong \mathbf{R}^\times \times \mathbf{R}^\times.$$

Then we see $S = \pi^{-1}(T)$ for some 1-dimensional closed subgroup T of $\mathbf{R}^\times \times \mathbf{R}^\times$. Thus we obtain $S = S(\alpha, \beta, \gamma)$ for some $(\alpha, \beta) \neq (0,0)$ and $\gamma \geq 0$. This is the case (d).

Finally, we assume $S_1 = N(2)$ and $S_{(1)} = N(2)^0$. Then we see

$$\dim S_2 = \dim S_{(2)} = 1 \text{ or } 2.$$

If $\dim S_2 = 2$, then S is the group listed in (a). Suppose $\dim S_2 = 1$. Then the identity component S_2^0 is conjugate with $L(2), \mathbf{SO}(2)$ or $D(2)^0$, up to conjugation, where

$$D(2) = \left\{ \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \right\}.$$

Since $N\mathbf{SO}(2) = \mathbf{SO}(2)$, the case $S_2^0 = \mathbf{SO}(2)$ is omitted. Suppose $S_2^0 = D(2)^0$. Then we see $S_2 = D(2)$ and $S_{(2)} = D(2)^0$, by (1). This is the case (c). Suppose $S_2^0 = L(2)$. Then we see

$$S = \left\{ \left(\begin{bmatrix} a & * \\ 0 & * \end{bmatrix}, \begin{bmatrix} \varepsilon b^k & * \\ 0 & * \end{bmatrix} \right) \mid a\varepsilon > 0,\ \varepsilon = \pm 1,\ k \in \mathbf{Z} \right\},$$

where $b \geq 1$ is a constant. This is a special type of (d). q.e.d.

Lemma 2.2. *Let $S(\alpha, \beta, \gamma)$ be the closed subgroup of G listed in* Lemma 2.1 (d). *If $\gamma \neq 0$, then $S(\alpha, \beta, \gamma)$ is not realized as an isotropy subgroup at some point of S^3 with respect to any extended G-action of the standard K-action ψ_0 of the first type.*

Proof. First we shall show that the restricted K-action on the quotient $G/S(\alpha, \beta, \gamma)$ has only one isotropy type $\Delta C_2 = \{(\varepsilon I_2, \varepsilon I_2) | \varepsilon = \pm 1\}$. Since $S(\alpha, \beta, \gamma)$ is a subgroup of $N(2) \times N(2)$, we obtain a natural projection $\pi : G/S(\alpha, \beta, \gamma) \to G/(N(2) \times N(2))$. The restricted K-action on $G/(N(2) \times N(2))$ has only one isotropy type $C_2 \times C_2$. Since π is K-equivariant, we see that the restricted K-action on the quotient space $G/S(\alpha, \beta, \gamma)$ has only one isotropy type ΔC_2.

By this result, we see that the quotient space $G/S(\alpha, \beta, \gamma)$ is open 3-manifold, if $S(\alpha, \beta, \gamma)$ is realized as an isotropy subgroup at some point of S^3 with respect to an extended G-action of the standard K-action ψ_0.

Now we shall show that $G/S(\alpha, \beta, \gamma)$ is a compact 3-manifold, if $\gamma \neq 0$. Notice that $L(2) \times L(2)$ is a normal subgroup of $S(\alpha, \beta, \gamma)$ and there is a natural G-equivariant diffeomorphism

$$G/(L(2) \times L(2)) \cong \mathbf{R}_0^2 \times \mathbf{R}_0^2.$$

Here $\mathbf{R}_0^2 = \mathbf{R}^2 - \{0\}$ and the action of $\mathbf{SL}(2, \mathbf{R})$ on $\mathbf{R}_0^2$ is canonical. The right action of $S(\alpha, \beta, \gamma)/(L(2) \times L(2))$ on $G/(L(2) \times L(2))$ corresponds to certain scalar multiplication on $\mathbf{R}_0^2 \times \mathbf{R}_0^2$. In fact, the right action of the coset represented by

$$\left(\begin{bmatrix} \varepsilon e^{\alpha t - \beta\gamma k} & * \\ 0 & * \end{bmatrix}, \begin{bmatrix} \varepsilon e^{\beta t + \alpha\gamma k} & * \\ 0 & * \end{bmatrix} \right) \quad ; \; t \in \mathbf{R}, \; k \in \mathbf{Z}$$

corresponds to the scalar multiplication

$$(\mathbf{u}, \mathbf{v}) \to (\varepsilon e^{\alpha t - \beta\gamma k}\mathbf{u}, \varepsilon e^{\beta t + \alpha\gamma k}\mathbf{v}).$$

Consider the natural diffeomorphism

$$S^1 \times S^1 \times \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_0^2 \times \mathbf{R}_0^2$$

defined by $(\mathbf{u}, \mathbf{v}, \lambda, \mu) \to (\lambda\mathbf{u}, \mu\mathbf{v})$. Here, $\mathbf{R}_+$ is the set of all positive real numbers. Then the above scalar multiplication corresponds to the following :

$$(\mathbf{u}, \mathbf{v}, \lambda, \mu) \to (\varepsilon\mathbf{u}, \varepsilon\mathbf{v}, e^{\alpha t - \beta\gamma k}\lambda, e^{\beta t + \alpha\gamma k}\mu).$$

Hence we see that for each $\mathbf{u}, \mathbf{v} \in S^1$, the curve $f(s) = (\mathbf{u}, \mathbf{v}, e^{-\beta s}, e^{\alpha s})$ intersects transversely each $S(\alpha, \beta, \gamma)/(L(2) \times L(2))$-orbit through $(\mathbf{u}, \mathbf{v}, \lambda, \mu)$. Therefore, we obtain a diffeomorphism

$$G/S(\alpha, \beta, \gamma)^0 \cong S^1 \times S^1 \times \mathbf{R}.$$

If $\gamma = 0$, then $G/S(\alpha, \beta, 0)$ is K-equivariantly diffeomorphic to

$$(\mathbf{SO}(2) \times_{\Delta C_2} \mathbf{SO}(2)) \times \mathbf{R}.$$

But if $\gamma \neq 0$, then $G/S(\alpha, \beta, \gamma)$ is K-equivariantly diffeomorphic to

$$(\mathbf{SO}(2) \times_{\Delta C_2} \mathbf{SO}(2)) \times S^1,$$

and hence $G/S(\alpha, \beta, \gamma)$ is a compact 3-manifold. q.e.d.

Lemma 2.3. *Let S be a closed subgroup of G. Suppose S satisfies the following conditions.*

(1) $S \cap K = \{(I_2, I_2)\}, \; \mathbf{SO}(2) \times \{I_2\}$ *or* $\{I_2\} \times \mathbf{SO}(2)$,
(2) $\dim S \geq 3$.

Then, S is conjugate with one of the followings :

(a) $\mathbf{SL}(2,\mathbf{R}) \times \{I_2\}$, $\{I_2\} \times \mathbf{SL}(2,\mathbf{R})$,

(b) $\mathbf{SL}(2,\mathbf{R}) \times N(2)^0$, $N(2)^0 \times \mathbf{SL}(2,\mathbf{R})$,

(c) $\mathbf{SL}(2,\mathbf{R}) \times \left\{ \begin{bmatrix} e^{\gamma k} & * \\ 0 & * \end{bmatrix} \mid k \in \mathbf{Z} \right\}$, $\left\{ \begin{bmatrix} e^{\gamma k} & * \\ 0 & * \end{bmatrix} \mid k \in \mathbf{Z} \right\} \times \mathbf{SL}(2,\mathbf{R})$,

(d) $\mathbf{SO}(2) \times N(2)^0$, $N(2)^0 \times \mathbf{SO}(2)$,

(e) $N(2)^0 \times N(2)^0$,

(f) $T(\alpha,\beta,\gamma) = \left\{ \left(\begin{bmatrix} e^{\alpha t - \beta\gamma k} & * \\ 0 & * \end{bmatrix}, \begin{bmatrix} e^{\beta t + \alpha\gamma k} & * \\ 0 & * \end{bmatrix} \right) \mid t \in \mathbf{R}, k \in \mathbf{Z} \right\}$

for $(\alpha,\beta) \neq (0,0)$ and $\gamma \geq 0$.

Proof. Consider the case $S \cap K = \{(I_2, I_2)\}$ and the remainig case, respectively. The proof is quite similar to the one of Lemma 2.1. So we omit the details. q.e.d.

Lemma 2.4. *Let S be the closed subgroup of G listed in* Lemma 2.3. *The subgroups (a),(c),(d) and (f) for $\gamma \neq 0$ are not realized as an isotropy subgroup at some point of S^3 with respect to any extended G-action of the standard K-action φ_0 of the second type.*

Proof. Considering the isotropy types of the natural K-action on the coset space G/S, we can delete $(a), (c), (d)$. Moreover, we see

$$G/T(\alpha,\beta,0) \cong K \times \mathbf{R}, \quad G/T(\alpha,\beta,\gamma) \cong K \times S^1 \ (\gamma \neq 0)$$

as K-manifolds. So we can delete (f) for $\gamma \neq 0$. q.e.d.

Lemma 2.5. *For any extended continuous G-action of the standard K-action φ_0 of the second type, the two singular orbits of the K-action φ_0 are G-invariant.*

Proof. This is the direct consequence of Lemma 2.4. q.e.d.

3 Certain smooth actions of $\mathbf{SL}(2,\mathbf{R})\times\mathbf{SL}(2,\mathbf{R})$ on 3-manifolds, I

3.1. Let ψ be the standard action of $G = \mathbf{SL}(2,\mathbf{R}) \times \mathbf{SL}(2,\mathbf{R})$ of the first type on S^3. Then the isotropy subgroup at E_{12} coincides with $N_{2,2}$. The orbit through E_{12} is the only one compact orbit.

The isotropy subgroups at $I_2/\sqrt{2}$ and $J/\sqrt{2}$ coincide with

$$\{(A,A) | A \in \mathbf{SL}(2,\mathbf{R})\} \text{ and } \{(A, JAJ^{-1}) | A \in \mathbf{SL}(2,\mathbf{R})\},$$

respectively. The orbits through these points are just two open orbits. Put

$$S^3_+ = \{X \in S^3 \mid \det X \geq 0\}, \quad S^3_- = \{X \in S^3 \mid \det X \leq 0\}.$$

These are closed invariant subsets. Put

$$\begin{aligned} \mathbf{D}^2_+ &= \left\{ c(a,b) \begin{bmatrix} 1+a & b \\ b & 1-a \end{bmatrix} \mid a^2+b^2 \leq 1 \right\} \\ &= \{c(\lambda)(I_2 + \lambda J R(\theta)) \mid \theta \in \mathbf{R}, 0 \leq \lambda \leq 1\}, \end{aligned}$$

where $c(a,b) = 1/\sqrt{2(1+a^2+b^2)}$ and $c(\lambda) = 1/\sqrt{2(1+\lambda^2)}$. This is a closed subset of S^3_+ and there are K-equivariant diffeomorphisms :

$$K \times_{\Delta \mathbf{SO}(2)} \mathbf{D}^2_+ \xrightarrow{f} \mathbf{SO}(2) \times \mathbf{D}^2_+ \xrightarrow{g} S^3_+.$$

Here, $K \times_{\Delta \mathbf{SO}(2)} \mathbf{D}^2_+$ is the quotient manifold of $K \times \mathbf{D}^2_+$ by the equivalence relation

$$((A,B),X) \sim ((AC^{-1}, BC^{-1}), CXC^{-1}) \ \text{ for } \ C \in \mathbf{SO}(2)$$

and the mappings f, g are defined by

$$f([(A,B),X]) = (AB^{-1}, BXB^{-1}), \quad g(A,X) = AX.$$

Moreover the K-action on $\mathbf{SO}(2) \times \mathbf{D}^2_+$ is defined by

$$(A,B) \cdot (C,X) = (ACB^{-1}, BXB^{-1}).$$

Put

$$\begin{aligned} \mathbf{D}^2_- &= \left\{ c(a,b) \begin{bmatrix} a+1 & -b \\ b & a-1 \end{bmatrix} \mid a^2+b^2 \leq 1 \right\} \\ &= \{c(\lambda)(J + \lambda R(\theta)) \mid \theta \in \mathbf{R}, 0 \leq \lambda \leq 1\}, \end{aligned}$$

where $c(a,b) = 1/\sqrt{2(1+a^2+b^2)}$ and $c(\lambda) = 1/\sqrt{2(1+\lambda^2)}$. This is a closed subset of S^3_- and there are K-equivariant diffeomorphisms :

$$K \times_{\Delta_J \mathbf{SO}(2)} \mathbf{D}^2_- \xrightarrow{f'} \mathbf{SO}(2) \times \mathbf{D}^2_- \xrightarrow{g'} S^3_-.$$

Here, $K \times_{\Delta_J \mathbf{SO}(2)} \mathbf{D}^2_-$ is the quotient manifold of $K \times \mathbf{D}^2_-$ by the equivalence relation

$$((A,B),X) \sim ((AC^{-1}, BC), CXC^{-1}) \ \text{ for } \ C \in \mathbf{SO}(2)$$

and the mappings f', g' are defined by

$$f'([(A,B),X]) = (AB, B^{-1}XB^{-1}), \quad g'(A,X) = AX.$$

Moreover the K-action on $\mathbf{SO}(2)\times\mathbf{D}^2_-$ is defined by

$$(A,B)\cdot(C,X) = (ABC, B^{-1}XB^{-1}).$$

3.2. Let $\psi_{(\alpha,\beta)}$ be the G-action on $(S^1\times_{\Delta C_2} S^1)\times S^1$ defined by

$$\begin{aligned}&\psi_{(\alpha,\beta)}((A,B),(\mathbf{u},\mathbf{v},(\lambda,\mu)))\\ &\quad = (\|A\mathbf{u}\|^{-1}A\mathbf{u}, \|B\mathbf{v}\|^{-1}B\mathbf{v}, (c\lambda\|A\mathbf{u}\|^{\alpha}\|B\mathbf{v}\|^{\beta}, c\mu\|A\mathbf{u}\|^{-\alpha}\|B\mathbf{v}\|^{-\beta})),\end{aligned}$$

where c is a positive real number. Here α,β are fixed real numbers such that $(\alpha,\beta)\neq(0,0)$ and $S^1\times_{\Delta C_2} S^1$ is the quotient manifold of $S^1\times S^1$ by the equivalence relaion $(\mathbf{u},\mathbf{v})\sim(-\mathbf{u},-\mathbf{v})$. Then the isotropy subgroup at $(\mathbf{e}_1,\mathbf{e}_1,(\lambda,\mu))$ coincides with

$$N_{2,2} \text{ for } \lambda\mu = 0 \ \text{ and } \ S(-\beta,\alpha,0) \text{ for } \lambda\mu\neq 0,$$

respectively. Put

$$M_{(\alpha,\beta)} = (S^1\times_{\Delta C_2} S^1)\times S^1_+,$$

where $S^1_+ = \{(\lambda,\mu)\in S^1 \mid \lambda\geq 0, \mu\geq 0\}$. This is an invariant closed subset.

3.3. Let ψ_{1D} be the G-action on $T^3/_\sim$ defined by

$$\psi_{1D}((A,B),(\mathbf{u},\mathbf{v},\mathbf{w})) = (\|A\mathbf{u}\|^{-1}A\mathbf{u}, \|B\mathbf{v}\|^{-1}B\mathbf{v}, \|B\mathbf{w}\|^{-1}B\mathbf{w}),$$

where $T^3/_\sim$ is the quotient manifold of $T^3 = S^1\times S^1\times S^1$ by the equivalence relation $(\mathbf{u},\mathbf{v},\mathbf{w})\sim(-\mathbf{u},-\mathbf{v},-\mathbf{w})$. Then the isotropy subgroup at $(\mathbf{e}_1,\mathbf{e}_1,\mathbf{e}_2)$ coincides with

$$\left\{\left(\begin{bmatrix} a & * \\ 0 & * \end{bmatrix}, \begin{bmatrix} b & 0 \\ 0 & * \end{bmatrix}\right) \mid ab>0\right\}.$$

The isotropy subgroup at $(\mathbf{e}_1,\mathbf{e}_1,\mathbf{e}_1)$ coincides with $N_{2,2}$. Put

$$M_{1D} = \{(\mathbf{u},\mathbf{v},\mathbf{w}) \mid \det[\mathbf{v},\mathbf{w}]\geq 0\}.$$

This is an invariant closed set.

Let ψ_{2D} be the G-action on $T^3/_\sim$ defined by

$$\psi_{2D}((A,B),(\mathbf{u},\mathbf{v},\mathbf{w})) = (\|A\mathbf{u}\|^{-1}A\mathbf{u}, \|A\mathbf{v}\|^{-1}A\mathbf{v}, \|B\mathbf{w}\|^{-1}B\mathbf{w}).$$

Then the isotropy subgroup at $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1)$ coincides with

$$\left\{ \left(\begin{bmatrix} a & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} b & * \\ 0 & * \end{bmatrix} \right) \mid ab > 0 \right\}.$$

The isotropy subgroup at $(\mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_1)$ coincides with $N_{2,2}$. Put

$$M_{2D} = \{(\mathbf{u}, \mathbf{v}, \mathbf{w}) \mid \det[\mathbf{u}, \mathbf{v}] \geq 0\}.$$

This is an invariant closed set.

Remark. $S^3_+, S^3_-, M_{(\alpha,\beta)}, M_{1D}$ and M_{2D} are compact 3-manifolds with smooth G-action, each of which has a boundary. S^3_+, S^3_- have a connected boundary, and the others have just two boundary components. Each boundary component of the above 3-manifolds is diffeomorphic to $G/N_{2,2}$ as a smooth G-manifold.

4 Continuous actions of $\mathbf{SL}(2, \mathbf{R}) \times \mathbf{SL}(2, \mathbf{R})$ on S^3 of the first type

In the previous section, we prepare the compact 3-manifolds S^3_+, S^3_-, $M_{(\alpha,\beta)}, M_{1D}$ and M_{2D} with smooth action of $G = \mathbf{SL}(2, \mathbf{R}) \times \mathbf{SL}(2, \mathbf{R})$. Consider the disjoint union

$$M_0 \cup M_1 \cup \cdots \cup M_r \cup M_{r+1}.$$

Here, $M_0 = S^3_+$, $M_{r+1} = S^3_-$ and each $M_i (i = 1, 2, \cdots, r)$ is one of $M_{(\alpha,\beta)}$, M_{1D} and M_{2D}, respectively. Put M the space obtained from the above disjoint union pasting the boundaries one after another by equivariant G-diffeomorphisms. Then, we see M is naturally diffeomorphic to S^3 with the standard K-action of the first type. Moreover, M has the natural continuous G-action. In such a way, we obtain an extended continuous G-action of the standard K-action of the first type on S^3. In consequence, we obtain the following result.

Theorem 4.1. *For each positive integer r, there exist uncountably many topologically distinct continuous $\mathbf{SL}(2, \mathbf{R}) \times \mathbf{SL}(2, \mathbf{R})$ actions on S^3, which are extensions of the standard $\mathbf{SO}(2) \times \mathbf{SO}(2)$ action of the first type and the number of open orbits of which is $r + 2$.*

Considering the differentiability of the above G-actions, we obtain the following result.

Theorem 4.2. *Each action of $\mathbf{SL}(2, \mathbf{R}) \times \mathbf{SL}(2, \mathbf{R})$ on S^3 is not C^2-differentiable for any decomposition* :

$$S^3 \cong M_0 \cup M_1 \cup \cdots \cup M_r \cup M_{r+1} \ (r \geq 1).$$

Proof. We shall show that G-action on $M_0 \cup M_1$ is not C^2- differentiable. First, we introduce local coordinates (τ,θ,λ) on $S^3_+, M_{(\alpha,\beta)}, M_{1D}$ and M_{2D}, as follows. Put

$$X = X(\tau,\theta,\lambda) = \frac{1}{\sqrt{2(1+\lambda^2)}}R(\tau)(I_2+\lambda J)R(\theta)^{-1},$$

where $0<\lambda\le 1$. Then $X\in S^3_+$ and $X(0,0,1)=E_{11}$. This is the local coordinates on S^3_+. Suppose

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = X(\tau,\theta,\lambda).$$

Then we obtain the following relations :

$$\lambda = \frac{\sqrt{1-4(ad-bc)^2}}{1+2(ad-bc)},\ \tan 2\tau = \frac{-2(ac+bd)}{a^2+b^2-c^2-d^2},\ \tan 2\theta = \frac{-2(ab+cd)}{a^2-b^2+c^2-d^2}.$$

Moreover, we obtain the following relations :

$$\tan(\tau-\theta) = \frac{b-c}{a+d},\quad \tan(\tau+\theta) = \frac{b+c}{d-a}.$$

Put

$$Y_0(\tau,\theta,\lambda) = (R(\tau)\mathbf{e}_1, R(\theta)\mathbf{e}_2, (\lambda,\sqrt{1-\lambda^2})),\ 0\le\lambda\le 1.$$

This is the local coordinates on $M_{(\alpha,\beta)}$. Put

$$\begin{aligned} Y_1(\tau,\theta,\lambda) &= (R(\tau)\mathbf{e}_1, R(\theta)(\lambda\mathbf{e}_1+\mathbf{e}_2)/\sqrt{1+\lambda^2}, R(\theta)\mathbf{e}_2), \\ Y_2(\tau,\theta,\lambda) &= (R(\tau)\mathbf{e}_1, R(\tau)(\mathbf{e}_1+\lambda\mathbf{e}_2)/\sqrt{1+\lambda^2}, R(\theta)\mathbf{e}_2). \end{aligned}$$

These are the local coordinates on M_{1D} and M_{2D}, respectively.

Next, we consider the tangent vector fields on these manifolds corresponding to the one-parameter group $\{g(t)\mid t\in\mathbf{R}\}$, where

$$g(t) = \left(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right).$$

Put

$$X(\tau(t),\theta(t),\lambda(t)) = \psi(g(t), X(\tau,\theta,\lambda)).$$

Then we see

$$\begin{aligned} \dot\tau(0) &= (1+\lambda^{-1}\sin(\tau+\theta)\sin(\tau-\theta) - \lambda\cos(\tau+\theta)\cos(\tau-\theta))/2, \\ \dot\theta(0) &= (1+\lambda^{-1}\sin(\tau+\theta)\sin(\tau-\theta) + \lambda\cos(\tau+\theta)\cos(\tau-\theta))/2, \\ \dot\lambda(0) &= (1-\lambda^2)(\sin 2\theta - \sin 2\tau)/2. \end{aligned}$$

The tangent vector field on S^3_+ can be described by these data. Put

$$Y_0(\tau(t), \theta(t), \lambda(t)) = \psi_{(\alpha,\beta)}(g(t), Y_0(\tau, \theta, \lambda)).$$

Then we see

$$\dot{\tau}(0) = \sin^2\tau, \quad \dot{\theta}(0) = \cos^2\theta, \quad \dot{\lambda}(0) = \lambda(\lambda^2 - 1)(\alpha \sin 2\tau - \beta \sin 2\theta).$$

The tangent vector field on $M_{(\alpha,\beta)}$ can be described by these data. Put

$$Y_1(\tau(t), \theta(t), \lambda(t)) = \psi_{1D}(g(t), Y_1(\tau, \theta, \lambda)).$$

Then we see

$$\dot{\tau}(0) = \sin^2\tau, \quad \dot{\theta}(0) = \cos^2\theta, \quad \dot{\lambda}(0) = -\lambda^2(1 + \sin 2\theta)/(1 + \lambda \sin 2\theta).$$

The tangent vector field on M_{1D} can be described by these data. Put

$$Y_2(\tau(t), \theta(t), \lambda(t)) = \psi_{2D}(g(t), Y_2(\tau, \theta, \lambda)).$$

Then we see

$$\dot{\tau}(0) = \sin^2\tau, \quad \dot{\theta}(0) = \cos^2\theta, \quad \dot{\lambda}(0) = \lambda(\sin 2\tau - \lambda \cos 2\tau).$$

The tangent vector field on M_{2D} can be described by these data.

Now we introduce new local coordinates (τ_1, θ_1, μ) on S^3_+ by

$$X_1(\tau_1, \theta_1, \mu) = X(\tau_1 + a(\mu), \theta_1 + b(\mu), c(\mu)).$$

Here, $a(\mu), b(\mu), c(\mu)$ are smooth functions on $0 < \mu \leq 1$ satisfying $a(1) = b(1) = 0, c(1) = 1$ and $c'(1) > 0$. Put

$$X_1(\tau_1(t), \theta_1(t), \mu(t)) = \psi(g(t), X_1(\tau_1, \theta_1, \mu)).$$

Then we see

$$\dot{\tau}_1(0) = \dot{\tau}(0) - a'(\mu)\dot{\mu}(0), \ \dot{\theta}_1(0) = \dot{\theta}(0) - b'(\mu)\dot{\mu}(0), \ \dot{\mu}(0) = (c^{-1})'(c(\mu))\dot{\lambda}(0).$$

In particular, we see

$$\dot{\mu}(0) = 0 \ (\mu = 1), \quad \frac{\partial}{\partial \mu}(\dot{\mu}(0)) = \sin 2\tau_1 - \sin 2\theta_1 \ (\mu = 1).$$

Hence we obtain

$$\frac{\partial}{\partial \mu}(\dot{\tau}_1(0)) = \frac{-c'(1)}{2} \neq 0 \ \text{ at}(\tau_1, \theta_1, \mu) = (0, 0, 1).$$

On the other hand,

$$\frac{\partial}{\partial \lambda}(\dot{\tau}(0)) = 0$$

for $M_{(\alpha,\beta)}, M_{1D}$ and M_{2D}, respectively.

This fact shows the G-action on $S^3_+ \cup M_1$ is not C^2-differentiable for $M_1 = M_{(\alpha,\beta)}, M_{1D}$ or M_{2D}. q.e.d.

5 Certain smooth actions of $\mathbf{SL}(2,\mathbf{R})\times\mathbf{SL}(2,\mathbf{R})$ on 3-manifolds, II

5.1. Considering the stereographic projection, we obtain the smooth action ϕ of $\mathbf{SL}(2,\mathbf{R})$ on $\mathbf{R}^2$ as follows :

$$\phi(A,X)=\frac{2}{1-\|X\|^2+\sqrt{(1-\|X\|^2)^2+4\|AX\|^2}}AX.$$

The origin is invariant and the other invariant sets are the followings :

$$\{X\in\mathbf{R}^2 \mid 0<\|X\|<1\},\quad \{X\in\mathbf{R}^2 \mid \|X\|=1\},\quad \{X\in\mathbf{R}^2 \mid \|X\|>1\}.$$

Put $M_+=S^1\times\mathbf{D}^2$ and $M_-=\mathbf{D}^2\times S^1$ and define

$$\begin{aligned}\phi_+((A,B),(\mathbf{u},\mathbf{v}))&=(\|A\mathbf{u}\|^{-1}A\mathbf{u},\phi(B,\mathbf{v})),\\ \phi_-((A,B),(\mathbf{u},\mathbf{v}))&=(\phi(A,\mathbf{u}),\|B\mathbf{v}\|^{-1}B\mathbf{v}).\end{aligned}$$

Then ϕ_+ is a smooth G-action on M_+ and ϕ_- is a smooth G-action on M_-.

The isotropy subgroup at $(\mathbf{e}_1,0)\in M_+$ coincides with $N(2)^0\times\mathbf{SL}(2,\mathbf{R})$, the one at $(0,\mathbf{e}_1)\in M_-$ coincides with $\mathbf{SL}(2,\mathbf{R})\times N(2)^0$, and the one at $(\mathbf{e}_1,\mathbf{e}_1)$ coincides with $N(2)^0\times N(2)^0$ for M_+ and M_-.

5.2. Let $\varphi_{(\alpha,\beta)}$ be the G-action on $T^3=S^1\times S^1\times S^1$ defined by

$$\begin{aligned}&\varphi_{(\alpha,\beta)}((A,B),(\mathbf{u},\mathbf{v},(\lambda,\mu)))\\ &\quad=(\|A\mathbf{u}\|^{-1}A\mathbf{u},\|B\mathbf{v}\|^{-1}B\mathbf{v},(c\lambda\|A\mathbf{u}\|^{\alpha}\|B\mathbf{v}\|^{\beta},c\mu\|A\mathbf{u}\|^{-\alpha}\|B\mathbf{v}\|^{-\beta})),\end{aligned}$$

where c is a positive real number. Here α,β are fixed real numbers such that $(\alpha,\beta)\neq(0,0)$.

The isotropy subgroup at $(\mathbf{e}_1,\mathbf{e}_1,(\lambda,\mu))$ coincides with

$$N(2)^0\times N(2)^0 \text{ for } \lambda\mu=0 \quad\text{and}\quad T(-\beta,\alpha,0) \text{ for } \lambda\mu\neq 0,$$

respectively. Put

$$M_{(\alpha,\beta)}=S^1\times S^1\times S^1_+,$$

where $S^1_+=\{(\lambda,\mu)\in S^1 \mid \lambda\geq 0,\mu\geq 0\}$. This is an invariant closed subset.

Remark. M_+,M_- and $M_{(\alpha,\beta)}$ are compact 3-manifolds with smooth G-action, each of which has a boundary. M_+,M_- have a connected boundary and $M_{(\alpha,\beta)}$ has just two boundary components. Each boundary component of the above 3-manifolds is diffeomorphic to $G/(N(2)^0\times N(2)^0)$ as a smooth G-manifold.

5.3. Let ξ be a smooth $\mathbf{R}^2$-action on $\mathbf{R}$. Now, we shall define a smooth G-action $\tilde{\xi}$ on $S^1 \times S^1 \times \mathbf{R}$ by ξ. Let

$$\rho : \mathbf{SO}(2) \times D(2)^0 \times L(2) \to \mathbf{SL}(2, \mathbf{R})$$

be the smooth bijection defined by the matrix multiplication $\rho(x, y, z) = xyz$. The inverse mapping of ρ defines a smooth decomposition of $\mathbf{SL}(2, \mathbf{R})$ to $\mathbf{SO}(2) \times D(2)^0 \times L(2)$. Put

$$D(\alpha) = \begin{bmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{bmatrix}.$$

We identify $\mathbf{R}^2$ with the following subgroup of G.

$$\{(D(\alpha), D(\beta)) \mid \alpha, \beta \in \mathbf{R}\}.$$

Define

$$\tilde{\xi}((A, B), (R(\tau)\mathbf{e}_1, R(\theta)\mathbf{e}_1, t)) = (R(\tau')\mathbf{e}_1, R(\theta')\mathbf{e}_1, \xi((\alpha, \beta), t)).$$

Here $A, B \in \mathbf{SL}(2, \mathbf{R})$ and

$$AR(\tau) = R(\tau')D(\alpha)L_1, \quad BR(\theta) = R(\theta')D(\beta)L_2$$

for some $L_1, L_2 \in L(2)$. Then we see $\tilde{\xi}$ is a well-defined smooth G-action on $S^1 \times S^1 \times \mathbf{R}$.

6 Continuous actions of $\mathbf{SL}(2, \mathbf{R}) \times \mathbf{SL}(2, \mathbf{R})$ on S^3 of the second type

In the previous section, we prepare the compact 3-manifolds M_+, M_- and $M_{(\alpha,\beta)}$ with smooth action of $G = \mathbf{SL}(2, \mathbf{R}) \times \mathbf{SL}(2, \mathbf{R})$. Consider the disjoint union

$$M_0 \cup M_1 \cup \cdots \cup M_r \cup M_{r+1}.$$

Here, $M_0 = M_+, M_{r+1} = M_-$ and $M_i = M_{(\alpha_i, \beta_i)} (i = 1, 2, \cdots, r)$. Put M the space obtained from the above disjoint union pasting the boundaries one after another by equivariant G-diffeomorphisms. Then, we see M is naturally diffeomorphic to S^3 with the standard K-action of the second type. Moreover, M has the natural continuous G-action. In such a way, we obtain an extended continuous G-action of the standard K-action of the second type on S^3. In consequence, we obtain the following result.

Theorem 6.1. *For each positive integer r, there exist uncountably many topologically distinct continuous $\mathbf{SL}(2, \mathbf{R}) \times \mathbf{SL}(2, \mathbf{R})$ actions on S^3, which* *are extensions of the standard* $\mathbf{SO}(2)\times\mathbf{SO}(2)$ *action of the second type and the number of open orbits of which is* $r+2$.

Next, we shall construct smooth G-actions on S^3, which are extensions of the standard K-action of the second type.

Let $\sigma(t)$ be a smooth real valued function such that

$$\begin{aligned} \sigma(t) = 1 &\text{ for } |t| \le 1, \\ \sigma(t) = 0 &\text{ for } |t| \ge 2, \\ 0 < \sigma(t) < 1 &\text{ for } 1 < |t| < 2. \end{aligned}$$

Define a tangent vector field Ξ_r on $\mathbf{R}$ by

$$\Xi_r = \sum_{i=0}^{r+1} \varepsilon_i \sigma(t+2-5i)\frac{d}{dt}.$$

Here $\varepsilon_0 = -1, \varepsilon_{r+1} = 1$ and $\varepsilon_i = \pm 1$ for $i = 1, 2, \cdots, r$. The support of the vector field Ξ_r is contained in the closed interval $[-4, 5r+5]$.

Let ϕ_r be the one-parameter group on $\mathbf{R}$ corresponding to the vector field Ξ_r. Now we define a smooth $\mathbf{R}^2$-action ξ_r on the open interval $(-4, 5r+5)$ by

$$\xi_r((s,t),x) = \begin{cases} \phi_r(t,x) \text{ for } -4 < x < 0 \\ \phi_r(s,x) \text{ for } 5r+1 < x < 5r+5 \\ \phi_r(a_i s + b_i t, x) \text{ for } 5i-4 < x < 5i \ (i = 1, 2, \cdots, r). \end{cases}$$

Here, a_i, b_i are given real numbers such that $(a_i, b_i) \neq (0,0)$. The smoothness of ξ_r is assured by the existence of the neutral zones.

By 5.3, we obtain a smooth G-action $\tilde{\xi}_r$ on $\tilde{M}_r = S^1 \times S^1 \times (-4, 5r+5)$ from ξ_r. Then the isotropy subgroup at $(\mathbf{e}_1, \mathbf{e}_1, -2)$ coincides with $N(2)^0 \times L(2)$, the one at $(\mathbf{e}_1, \mathbf{e}_1, 5r+3)$ coincides with $L(2) \times N(2)^0$ and the one at $(\mathbf{e}_1, \mathbf{e}_1, 5i+3)$ coincides with $T(-b_i, a_i, 0)$.

Put $\tilde{M}_+ = S^1 \times \mathbf{R}^2$ and $\tilde{M}_- = \mathbf{R}^2 \times S^1$ and define

$$\begin{aligned} \tilde{\phi}_+((A,B),(\mathbf{u},\mathbf{v})) &= (\|A\mathbf{u}\|^{-1}A\mathbf{u}, \phi(B,\mathbf{v})), \\ \tilde{\phi}_-((A,B),(\mathbf{u},\mathbf{v})) &= (\phi(A,\mathbf{u}), \|B\mathbf{v}\|^{-1}B\mathbf{v}). \end{aligned}$$

Then $\tilde{\phi}_+$ is a smooth G-action on $\tilde{M}_+$ and $\tilde{\phi}_-$ is a smooth G-action on $\tilde{M}_-$. Consider the disjoint union

$$\tilde{M}_+ \cup \tilde{M}_r \cup \tilde{M}_-.$$

Put $\tilde{M}$ the space obtained from the above disjoint union, identifying the open set $S^1 \times \{X \in \mathbf{R}^2 \mid \|X\| > 1\}$ of $\tilde{M}_+$ with the open set $S^1 \times S^1 \times (-4, 0)$

of $\tilde{M}_r$ by G-diffeomorphism and the open set $\{X \in \mathbf{R}^2 \mid \|X\| > 1\} \times S^1$ of $\tilde{M}_-$ with the open set $S^1 \times S^1 \times (5r+1, 5r+5)$ of $\tilde{M}_r$ by G-diffeomorphism. Then, $\tilde{M}$ has a smooth G-action. In consequence, we obtain the following result.

Theorem 6.2. *For each positive integer r, there exist uncountably many topologically distinct smooth $\mathbf{SL}(2,\mathbf{R}) \times \mathbf{SL}(2,\mathbf{R})$ actions on S^3, which are extensions of the standard $\mathbf{SO}(2) \times \mathbf{SO}(2)$ action of the second type and the number of open orbits of which is $r+4$.*

Remark. Let c be a positive real number. Define

$$\varphi_c((A,B), \mathbf{u} \oplus \mathbf{v}) = e^{\theta} A\mathbf{u} \oplus e^{c\theta} B\mathbf{v}.$$

Here, $\theta \in \mathbf{R}$ is determined by the condition

$$e^{2\theta}\|A\mathbf{u}\|^2 + e^{2c\theta}\|B\mathbf{v}\|^2 = 1.$$

Then, φ_c is a smooth action of $\mathbf{SL}(2,\mathbf{R}) \times \mathbf{SL}(2,\mathbf{R})$ on S^3, which is called the twisted linear action [4]. The isotropy subgroup at $(\mathbf{e}_1 \oplus \mathbf{e}_1)/\sqrt{2}$ coincides with $T(1,c,0)$.

7 Final remark

On smooth actions of non-compact semi-simple Lie groups on low dimensional manifolds, T.Asoh [1] and O.Yokoyama [9] have studied on smooth $\mathbf{SL}(2,\mathbf{C})$-actions on 3-manifolds, and K.Mukōyama [2] has studied on smooth $\mathbf{Sp}(2,\mathbf{R})$-actions on the 4-sphere. On the present article, we study on certain continuous $\mathbf{SL}(2,\mathbf{R}) \times \mathbf{SL}(2,\mathbf{R})$-actions on the 3-sphere.

Let G be a non-compact semi-simple Lie group and K be the maximal compact subgroup of G. On smooth G-actions on a sphere of which restricted K-action has principal orbits of codimension one, F.Uchida [5,6,7,8] has studied such actions for $G = \mathbf{SO}_0(p,q), \mathbf{Sp}(p,q)$ and $\mathbf{SL}(m,\mathbf{R}) \times \mathbf{SL}(n,\mathbf{R})$, and K.Mukōyama [3] has studied such actions for $G = \mathbf{SU}(p,q)$. On the present article, a complementary part of [8] is studied.

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